

Solutions in the generalized Proca theory with the nonminimal coupling to the Einstein tensor

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We investigate the static and spherically symmetric solutions in a class of the generalized Proca theory with the nonminimal coupling to the Einstein tensor. First, we show that the solutions in the scalar-tensor theory with the nonminimal derivative coupling to the Einstein tensor can be those in the generalized Proca theory with the vanishing field strength. We then show that when the field strength takes the nonzero value the static and spherically symmetric solutions can be found only for the specific value of the nonminimal coupling constant. Second, we investigate the first-order slow-rotation corrections to the static and spherically symmetric background. We find that for the background with the vanishing electric field strength the slowly rotating solution is identical to the Kerr- (anti-) de Sitter solutions in general relativity. On the other hand, for the background with the nonvanishing electric field strength the stealth property can be realized at the first order in the slow-rotation approximation.

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I. INTRODUCTION

Cosmological observations suggest the existence of the mysterious elements in the history of the Universe, such as the inflationary evolution at the beginning of the Universe, and dark matter and dark energy at the present day. General relativity (GR) with these new elements in the right-hand side of the Einstein equation as the energy-momentum sources are mathematically equivalent to the certain modification of GR where the left-hand side of the Einstein equation is modified by the new gravitational degrees of freedom in addition to the metric tensor [1, 2]. In many cases, the modification of GR can be described by a scalar-tensor theory of gravitation at least in a certain regime [3]. Realistic modification of GR should not contain the so-called Ostrogradsky ghosts associated with the higher-derivative interactions [4], and should be endowed with the mechanisms that could suppress the extra gravitational degrees of freedom around the locally gravitating sources [5, 6], in order to be compatible with the tests of GR in the weak gravity regime. Scalar-tensor theories that could satisfy all these requirements typically belong to the so-called generalized Galileon / Horndeski theory [7–10] where the equations of motion are given by the second-order differential equations despite the existence of the higher derivative interactions in the Lagrangian.

While the scalar-tensor Horndeski theory has been extensively investigated, it is also interesting to look for the similar theories for the other field species. In this paper, we consider a class of the generalized vector-tensor theories of gravitation where the equations of motion are given by the second-order differential equations. It was shown that if the gauge symmetry of the vector field is preserved, the Galileon-like extension of the vector field theory does not exist and only the Maxwell kinetic term is allowed [11]. A way out for this no-go theorem was to abandon the gauge symmetry. The introduction

of the mass term of the vector field breaks the gauge symmetry. In the vector field theory with the mass term $m^2 A_\mu A^\mu$, where A_μ is the vector field and the Greek indices (μ, ν, \dots) run the four-dimensional spacetime, the so-called Proca theory, the vector field contains the three propagating degrees of freedom, namely one longitudinal and two transverse degrees of freedom. The generalization of the massive vector field theory to the Galileon-like theory was first investigated in Refs. [12, 13], and then extended in Refs. [14–18] including the generalization of the interaction of the field strength with the double dual of the Riemann tensor Ref. [19]. In the generalized Proca theory, the screening mechanism and cosmology have been investigated in Refs. [16–18].

In this paper, we will investigate the static and spherically symmetric solutions in the subclass of the generalized Proca theory which possesses the nonminimal coupling of the vector field to the Einstein tensor $G^{\mu\nu} A_\mu A_\nu$, where $G_{\mu\nu}$ is the Einstein tensor associated with the metric $g_{\mu\nu}$. First, we will show that the solutions in the scalar-tensor Horndeski theory with the nonminimal derivative coupling to the Einstein tensor $G^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi$ [20–24] can also be those in the above generalized Proca theory with the vanishing field strength. In this subclass of the scalar-tensor Horndeski theory, the static and spherically symmetric solutions have been obtained in Refs. [25–28] (see also Refs. [29–31] for the more general theories and Ref. [32, 33] for the reviews), and the solutions particularly relevant for astrophysics or cosmology are the stealth Schwarzschild and the Schwarzschild- (anti-) de Sitter solutions which were originally obtained in Ref. [25].

On the other hand, the nonexistence of the black hole solutions with the massive vector field charge has been proven by Bekenstein in Refs. [34, 35]. As shown in recent work [36], however, the no-hair argument can be avoided once the nonminimal coupling $G^{\mu\nu} A_\mu A_\nu$ is introduced.¹ This cor-

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¹ As argued in Ref. [37], the no-hair argument for the Proca theory can also be circumvented for the complex massive Proca field.

responds to the simplest class of the ghost-free bilinear non-minimal couplings of the vector field to the divergence-free Lovelock tensors [38]. Moreover, as argued in Ref. [39], the nonminimal coupling of the vector field to the Einstein tensor, $G^{\mu\nu} A_\mu A_\nu$, can also arise from the quadratic gravitational theory in the Weyl geometry. References [36, 38] have obtained the black hole solutions for the particular value of the nonminimal coupling constant. Reference [40] has investigated the black hole solutions in the generalized Proca theory with the nonminimal coupling $RA^\mu A_\mu$, where R is the Ricci scalar curvature. In this paper, we will extend these former attempts in Refs. [36, 38], clarify the relations among the solutions, and also investigate the first-order slow-rotation corrections within the Hartle-Thorne approximation [41, 42] along the line of Refs. [43, 44].

The paper is organized as follows: In Sec. II, we will introduce the generalized Proca theory with the nonminimal coupling to the Einstein tensor and derive the equations of motion. In Sec. III, we will show how the solutions in the scalar-tensor theory can be described in the generalized Proca theory with the vanishing field strength. In Sec. IV, we will obtain the solutions with the Coulomb potential in the temporal component of the vector field. In Sec. V, we will explore the solutions with other forms of the vector field. In Sec. VI, we will investigate the first-order slow-rotation corrections to the static and spherically symmetric solutions. The last section, Sec. VII, is devoted to giving the concluding remarks.

II. THE GENERALIZED PROCA THEORY WITH THE NONMINIMAL COUPLING TO THE EINSTEIN TENSOR

In this paper, we consider the generalized Proca theory given by

$$S = \int d^4x \sqrt{-g} \left[\frac{m_p^2}{2} (R - 2\Lambda) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - (m^2 g_{\mu\nu} - \beta G_{\mu\nu}) A^\mu A^\nu \right], \quad (1)$$

where $g_{\mu\nu}$ is the metric tensor, R and $G_{\mu\nu}$ are the Ricci scalar and the Einstein tensor associated with $g_{\mu\nu}$, m_p and Λ are the reduced Planck mass and the cosmological constant, respectively. A_μ is the vector field, m and β are the mass and the (dimensionless) nonminimal coupling constant of the vector field, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength. After partially integrating and ignoring the boundary terms, the action (1) can be rewritten as

$$S = \int d^4x \sqrt{-g} \left[\frac{m_p^2}{2} (R - 2\Lambda) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - m^2 g_{\mu\nu} A^\mu A^\nu + \beta \left((\nabla_\mu A^\mu)^2 - \nabla_\mu A_\nu \nabla^\nu A^\mu - \frac{1}{2} A_\mu A^\mu R \right) \right]. \quad (2)$$

According to the formulation in Refs. [13, 16, 17], the action (2) corresponds to the case that

$$G_2(X) = 2m^2 X - \Lambda m_p^2, \quad G_4(X) = \frac{m_p^2}{2} + \beta X, \quad (3)$$

with $c_2 = 0$ and $G_3(X) = G_5(X) = 0$ where we have defined $X := -\frac{1}{2} g^{\mu\nu} A_\mu A_\nu$. The action (1) is quadratic in A_μ and hence reflection-symmetric under $A_\mu \rightarrow -A_\mu$. Thus if a set of the metric and the vector field $(g_{\mu\nu}, A_\mu)$ is a solution of the theory (2), the set $(g_{\mu\nu}, -A_\mu)$ is also a solution of it.

Varying the action (1) with respect to the metric $g_{\mu\nu}$ and the vector field A_μ , respectively, the Einstein equation and the vector field equation of motion are obtained by

$$\begin{aligned} m_p^2 (G_{\mu\nu} + \Lambda g_{\mu\nu}) &= \left(F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \right) + 2m^2 \left(A_\mu A_\nu - \frac{1}{2} g_{\mu\nu} A^\rho A_\rho \right) \\ &\quad + \beta (A^\rho A_\rho G_{\mu\nu} + A_\mu A_\nu R) \\ &\quad - \beta g_{\mu\nu} \left[(\nabla_\rho A^\rho)^2 - 2\nabla_\rho A_\sigma \nabla^\rho A^\sigma + \nabla_\rho A_\sigma \nabla^\sigma A^\rho - 2A_\rho \square A^\rho + 2A^\rho \nabla_\rho \nabla_\sigma A^\sigma \right] \\ &\quad - 2\beta \left[\nabla_\mu A_\rho \nabla_\nu A^\rho - \nabla_\rho A^\rho \nabla_{(\mu} A_{\nu)} - \nabla_\rho A_{(\mu} \nabla_{\nu)} A^\rho + \nabla_\rho A_\mu \nabla^\rho A_\nu + A_\rho \nabla_{(\mu} \nabla_{\nu)} A^\rho \right. \\ &\quad \left. - A^\rho \nabla_\rho \nabla_{(\mu} A_{\nu)} + A_{(\mu} \square A_{\nu)} - 2A_{(\mu} \nabla_{\nu)} \nabla_\sigma A^\sigma + A_{[\mu} \nabla_\rho \nabla_{\nu]} A^\rho \right], \quad (4a) \\ \nabla_\mu F^{\mu\nu} &= 2(m^2 g^{\mu\nu} - \beta G^{\mu\nu}) A_\mu. \quad (4b) \end{aligned}$$

As expected the equations of motion (4) are given by the second-order differential equations. Acting ∇_ν on Eq. (4b)

with $\nabla_\nu \nabla_\mu F^{\mu\nu} = 0$, we obtain

$$\nabla_\nu [(m^2 g^{\mu\nu} - \beta G^{\mu\nu}) A_\mu] = 0, \quad (5)$$

which gives the constraint relation among the four compo-

nents of the vector field A_μ , leaving the three physical degrees of freedom, namely one longitudinal and two transverse degrees of freedom.

III. FROM THE SCALAR-TENSOR THEORY TO THE GENERALIZED PROCA THEORY

In this section, we show how the static and spherically symmetric solutions in the scalar-tensor theory (8) are expressed in the generalized Proca theory (1) with the vanishing electric field strength.

A. The correspondence

We assume that the vector field A_μ can be decomposed into the part given by the gradient of the scalar function φ and the remaining vector field part B_μ ,

$$A_\mu = \partial_\mu \varphi + B_\mu. \quad (6)$$

Since obviously the scalar function does not contribute to the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu B_\nu - \partial_\nu B_\mu =: F_{\mu\nu}^{(B)}$, plugging Eq. (6) into the action (1), we obtain

$$S = \int d^4x \sqrt{-g} \left[\frac{m_p^2}{2} (R - 2\Lambda) - \frac{1}{4} F_{\mu\nu}^{(B)} F^{(B)\mu\nu} - (m^2 g_{\mu\nu} - \beta G_{\mu\nu}) (\nabla^\mu \varphi + B^\mu) (\nabla^\nu \varphi + B^\nu) \right]. \quad (7)$$

In the case of $B_\mu = 0$, namely when the vector field is given by the gradient of the scalar function $A_\mu = \partial_\mu \varphi$, the action (7) reduces to the shift-symmetric scalar-tensor theory with the nonminimal derivative coupling to the Einstein tensor,

$$S = \int d^4x \sqrt{-g} \left[\frac{m_p^2}{2} (R - 2\Lambda) - (m^2 g_{\mu\nu} - \beta G_{\mu\nu}) \nabla^\mu \varphi \nabla^\nu \varphi \right], \quad (8)$$

which involves the metric $g_{\mu\nu}$ and the scalar field φ as the physical degrees of freedom [20–24].

The static and spherically symmetric black hole solutions in the scalar-tensor theory (8) have been investigated in Refs. [25–28, 32, 33] under the static and spherically symmetric metric ansatz

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{h(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (9)$$

where t and r are the time and radial coordinates, θ and ϕ are the polar and azimuthal coordinates of the two-sphere, respectively, and $f(r)$ and $h(r)$ are the functions of only the radial coordinate r . In the static and spherically symmetric background (9), the vector field has the nonvanishing t and r components

$$A_\mu = (A_0(r), A_1(r), 0, 0), \quad (10)$$

where $A_0(r)$ and $A_1(r)$ are also only the functions of r . Because of the reflection symmetry of the generalized Proca theory (1) and the absence of the cross terms of $A_0(r)$ and $A_1(r)$ (and their derivatives) in the Einstein equation (4a) under the ansatz Eqs. (9) and (10), if a set $(A_0(r), A_1(r)) = (c_0(r), c_1(r))$ is a solution, other sets $(A_0(r), A_1(r)) = (c_0(r), -c_1(r)), (-c_0(r), c_1(r)), (-c_0(r), -c_1(r))$ are also solutions. In each case, among them we will show the two independent branches $(A_0(r), A_1(r)) = (c_0(r), \pm c_1(r))$.

We then derive the condition that the solution in the generalized Proca theory (1) is also the solution in the scalar-tensor theory (8) within the ansatz Eqs. (9) and (10). Imposing the condition that $B_\mu = 0$, only the nontrivial component of the field strength $F_{rt} = A'_0(r) = 0$, which by integration gives

$$A_0(r) = P, \quad (11)$$

where P is the constant. Then from Eq. (6) with $B_\mu = 0$, we identify

$$\partial_t \varphi = P, \quad \partial_r \varphi = A_1(r). \quad (12)$$

Further integrating Eq. (12), the scalar function φ is found to take the form of

$$\varphi(t, r) = P t + \psi(r), \quad \psi(r) := \int dr A_1(r). \quad (13)$$

The scalar function of the form (13) is exactly the same as in the black hole solutions found in the scalar-tensor theory (8). (See Ref. [25] for $P \neq 0$ and Refs. [26–28] for $P = 0$.) Thus Eq. (12) gives how to express the solution in the scalar-tensor theory (8) in the generalized Proca theory (1) with the vanishing field strength.

On the other hand, the solution with the nonconstant $A_0(r)$, giving rise to the nonvanishing electric field strength $F_{rt} = A'_0(r)$, does not contain the counterpart in the scalar-tensor theory (8). In the rest of this section, we focus on the case that $B_\mu = 0$ and show how the solutions in the scalar-tensor theory (8) discussed in Refs. [25–28] can be expressed in the generalized Proca theory (1).

B. The stealth Schwarzschild solution

The first example of the static and spherically symmetric solution in the scalar-tensor theory (8) is the stealth Schwarzschild solution obtained for $m = \Lambda = 0$ [25]. In the generalized Proca theory (1), for general P in Eq. (11) the solution is given by

$$f(r) = h(r) = 1 - \frac{2M}{r}, \quad (14a)$$

$$A_1(r) = \pm \sqrt{\frac{2M}{r}} \frac{P}{f}, \quad (14b)$$

where M is the integration constant that physically corresponds to the mass of the black hole. The parameter M appearing in the solutions discussed in the rest also has the same physical meaning.

This is the stealth black hole solution in the sense that the amplitude of the vector field P does not appear in the metric. Introducing the tortoise coordinate $dr_* = \frac{dr}{f}$,

$$A_\mu dx^\mu = P \left(dt \pm \sqrt{\frac{2M}{r}} dr_* \right), \quad (15)$$

which near the event horizons $r = 2M$ reduces to

$$A_\mu dx^\mu \approx P (dt \pm dr_*) = P \times \begin{cases} dv \\ du, \end{cases} \quad (16)$$

where $v := t + r_*$ and $u := t - r_*$ are the advanced and retarded null coordinates. The null coordinates v and u are regular at the future and past event horizons, respectively, ensuring the regularity of the scalar field there for each branch in the context of the scalar-tensor theory (8) [25, 29, 31].

C. The Schwarzschild- (anti-) de Sitter solution

Similarly, for $P = \pm \frac{m_p}{m} \sqrt{\frac{m^2 + \beta\Lambda}{2\beta}}$ in Eq. (11) the Schwarzschild- (anti-) de Sitter solution in the scalar-tensor theory (8) obtained in Ref. [25] is also expressed in the generalized Proca theory by

$$f(r) = h(r) = 1 - \frac{2M}{r} + \frac{m^2}{3\beta} r^2, \quad (17a)$$

$$A_1(r) = \pm \frac{m_p}{m} \sqrt{-\frac{(m^2 + \beta\Lambda)(m^2 r^3 - 6M\beta)}{6\beta^2 r}} \frac{1}{f(r)}, \quad (17b)$$

where the bare value of the cosmological constant Λ does not appear in the metric functions $f(r)$ and $h(r)$, and from the metric functions (17a) the effective cosmological constant is read as $\Lambda_{\text{eff}} = -\frac{m^2}{\beta}$. Thus the spacetime is either asymptotically de Sitter or anti- de Sitter for $\beta < 0$ and $\beta > 0$, respectively. The positivity inside the square root in $A_0(r)$ requires $\Lambda \geq -\frac{m^2}{\beta}$, irrespective of the sign of β . Thus for $\beta > 0$, Λ can be either positive or negative, while for $\beta < 0$, Λ is always positive. For $m^2 = -\beta\Lambda$, $A_1(r)$ vanishes and the solution (17) reduces to the Schwarzschild- anti- de Sitter in GR with the cosmological constant Λ ,

$$f(r) = h(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2. \quad (18)$$

Introducing the tortoise coordinate $dr_* = \frac{dr}{f}$,

$$A_\mu dx^\mu = \frac{m_p}{m} \sqrt{\frac{m^2 + \beta\Lambda}{2\beta}} \left(dt \pm \sqrt{\frac{-m^2 r^3 + 6M\beta}{3\beta r}} dr_* \right), \quad (19)$$

which near the (either event or cosmological) horizons $r \approx r_*$ satisfying $-m^2 r_*^3 + 6M\beta = 3\beta r_*$ becomes

$$A_\mu dx^\mu \approx \frac{m_p}{m} \sqrt{\frac{m^2 + \beta\Lambda}{2\beta}} \times \begin{cases} dv \\ du, \end{cases} \quad (20)$$

where v and u are defined as in Eq. (16). In the limit of $\beta \rightarrow -\frac{m^2}{\Lambda}$ the vector field trivially vanishes and the Schwarzschild- (anti-) de Sitter solution in GR with the cosmological constant Λ is recovered. The null coordinate v is regular at the future event and past cosmological (only for $\beta < 0$) horizons, while the null coordinate u is regular at the past event and future cosmological (only for $\beta < 0$) horizons, ensuring the regularity of the scalar field there for each branch in the context of the scalar-tensor theory (8) [25, 29, 31].

D. The asymptotically anti- de Sitter solution

Finally, for $P = 0$ in Eq. (11) where in the theory Eq. (8) the scalar field φ is time independent, the asymptotically anti- de Sitter solution obtained in Refs. [26–28] in the theory (8) is expressed in the generalized Proca theory by

$$f(r) = \frac{1}{3mr\beta(m^2 - \beta\Lambda)^2} [m^7 r^3 - 3mr\beta^3 \Lambda^2 + m^3 r \beta^2 \Lambda (-6 + r^2 \Lambda) + m^5 \beta (9r - 2r^3 \Lambda - 24M)] \quad (21a)$$

$$+ 3\beta^{\frac{3}{2}} (m^2 + \beta\Lambda)^2 \arctan\left(\frac{mr}{\sqrt{\beta}}\right)],$$

$$h(r) = \frac{(m^2 - \beta\Lambda)^2 (m^2 r^2 + \beta)^2}{m^4 (m^2 r^2 + \beta (2 - r^2 \Lambda))^2} f(r), \quad (21b)$$

$$A_1(r) = \pm \sqrt{-\frac{m^2 + \beta\Lambda}{2\beta (m^2 r^2 + \beta) h(r)}} m_p r. \quad (21c)$$

We require $\beta > 0$ so that the domain of r is given by $0 < r < \infty$. Then, in order for $A_1(r)$ to be real outside the event horizon $h(r) > 0$, from Eq. (21c) we find $\Lambda \leq -\frac{m^2}{\beta}$.

From the large- r limit of the metric functions,

$$f(r) \approx \frac{m^2 r^2}{3\beta} + \frac{3m^2 + \beta\Lambda}{m^2 - \beta\Lambda} + \mathcal{O}\left(\frac{1}{r}\right), \quad (22a)$$

$$h(r) \approx \frac{m^2 r^2}{3\beta} + \frac{7m^2 + \beta\Lambda}{3(m^2 - \beta\Lambda)} + \mathcal{O}\left(\frac{1}{r}\right), \quad (22b)$$

we find that the effective cosmological constant is given by $\Lambda_{\text{eff}} = -\frac{m^2}{\beta} < 0$, and hence the spacetime is asymptotically anti- de Sitter. For the parameters satisfying the above bound, the function $f(r)$ has a single root that corresponds to the position of the unique event horizon. For $\Lambda < \frac{m^2}{\beta}$, the point $r = \sqrt{\frac{2\beta}{\beta\Lambda - m^2}}$ is not the curvature singularity.

For $m^2 = -\beta\Lambda$ which for $\beta > 0$ requires $\Lambda < 0$, $A_1(r)$ vanishes and the solution (21) reduces to the Schwarzschild- anti- de Sitter in GR with the cosmological constant Λ , Eq. (18).

IV. THE CASE OF THE VECTOR FIELD WITH THE FORM OF THE COULOMB POTENTIAL

In this section, we consider the case that the temporal component of the vector field $A_0(r)$ is given by the Coulomb potential as well as the constant term P ,

$$A_0(r) = P + \frac{Q}{r}, \quad (23)$$

where the constant Q corresponds to the electric charge.

For $m = \beta = 0$ where the gauge symmetry is recovered, the Reissner-Nortström -(anti-) de Sitter solution is obtained by

$$f(r) = h(r) = 1 - \frac{\Lambda}{3}r^2 - \frac{2M}{r} + \frac{Q^2}{2m_p^2 r^2}, \quad (24a)$$

$$A_1(r) = 0. \quad (24b)$$

A. The stealth Schwarzschild solution

First, we consider the case of $m = 0$ and $\Lambda = 0$. As argued in Ref. [36], only for $\beta = \frac{1}{4}$ the stealth Schwarzschild solution (14) is obtained by

$$f(r) = h(r) = 1 - \frac{2M}{r}, \quad (25a)$$

$$A_1(r) = \pm \frac{\sqrt{Q^2 + 2P(Q + MP)r}}{r} \frac{1}{f(r)}. \quad (25b)$$

The positivity inside the square root (25b) for an arbitrary r requires $P(Q + MP) \geq 0$.

B. The Schwarzschild- (anti-) de Sitter solution

We then consider the case that $m^2 \neq 0$ and $\Lambda \neq 0$, where the generalization of the Schwarzschild- (anti-) de Sitter solution (17) is obtained only for $\beta = \frac{1}{4}$.

In the case of $m^2 > 0$, only for $P = \pm \frac{m_p}{\sqrt{2m}} \sqrt{4m^2 + \Lambda}$ in Eq. (23) the Schwarzschild- anti- de Sitter solution is obtained by

$$f(r) = h(r) = 1 - \frac{2M}{r} + \frac{4m^2}{3}r^2, \quad (26a)$$

$$A_1(r) = \pm \frac{1}{\sqrt{3mr}f(r)} \left[m_p^2 r (3M - 2m^2 r^3) (4m^2 + \Lambda) \pm 3mm_p r \sqrt{2(4m^2 + \Lambda)Q + 3m^2 Q^2} \right]^{\frac{1}{2}}. \quad (26b)$$

Similarly in the case of $m^2 < 0$, only for $P = \pm \frac{m_p}{\sqrt{2|m|}} \sqrt{4|m|^2 - \Lambda}$ in Eq. (23), the Schwarzschild- de Sitter

solution is obtained by

$$f(r) = h(r) = 1 - \frac{2M}{r} - \frac{4|m|^2}{3}r^2, \quad (27a)$$

$$A_1(r) = \pm \frac{1}{\sqrt{3|m|rf(r)}} \left[m_p^2 r (3M + 2|m|^2 r^3) (4|m|^2 - \Lambda) \pm 3|m|m_p r \sqrt{2(4|m|^2 - \Lambda)Q + 3|m|^2 Q^2} \right]^{\frac{1}{2}}. \quad (27b)$$

For the solution (26), in order for $A_0(r)$ to be real, we require $\Lambda \geq -4m^2$. For a large r , however, the combination inside the square root in Eq. (26b) always becomes negative, and hence the solution (26) may not be regarded as the physical one. On the other hand, for the solution (27), in order for $A_0(r)$ to be real, we require that $\Lambda < 4|m|^2$ and then the positivity inside the square root of Eq. (27b) between the event and cosmological horizons can be naturally realized.

C. The asymptotically anti- de Sitter solution

Finally, we consider the case of $P = 0$ in Eq. (23). As for the other cases, only for $\beta = \frac{1}{4}$, the asymptotically anti- de Sitter solution (21) is obtained by

$$f(r) = \frac{1}{6mr(\Lambda - 4m^2)^2} \times \left\{ -6\Lambda^2 mr + 128m^7 r^3 - 32m^5 (24M + 2\Lambda r^3 - 9r) + 8\Lambda m^3 r (\Lambda r^2 - 6) + 3(\Lambda + 4m^2)^2 \arctan(2mr) \right\}, \quad (28a)$$

$$h(r) = \frac{(\Lambda - 4m^2)^2 (4m^2 r^2 + 1)^2}{16m^4 (4m^2 r^2 - \Lambda r^2 + 2)^2} f(r), \quad (28b)$$

$$A_1(r) = \pm \frac{\sqrt{Q^2 (1 + 4m^2 r^2) - 2m_p^2 (\Lambda + 4m^2) r^4 f(r)}}{r \sqrt{f(r)h(r)} \sqrt{1 + 4r^2 m^2}}. \quad (28c)$$

In order for $A_1(r)$ to be real for $f(r) > 0$, $\Lambda + 4m^2 \leq 0$. From the large- r limit of the metric functions $f(r)$ and $h(r)$,

$$f(r) = \frac{4m^2}{3}r^2 + \frac{12m^2 + \Lambda}{4m^2 - \Lambda} + \mathcal{O}\left(\frac{1}{r}\right),$$

$$h(r) = \frac{4m^2}{3}r^2 + \frac{28m^2 + \Lambda}{12m^2 - 3\Lambda} + \mathcal{O}\left(\frac{1}{r}\right), \quad (29)$$

the effective cosmological constant is read as $\Lambda_{\text{eff}} = -4m^2 < 0$ and hence the spacetime is asymptotically anti- de Sitter. For $M > 0$, the function $f(r)$ has a single root which corresponds to the position of the unique event horizon. For $\Lambda \leq 4m^2$, the point $r = \sqrt{\frac{2}{\Lambda - 4m^2}}$ is not the curvature singularity.

For $m = \pm \frac{\sqrt{-\Lambda}}{2}$ where the positivity of m^2 requires $\Lambda < 0$, the solution (28) reduces to the Schwarzschild- anti- de Sitter

$$h(r) = f(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2,$$

$$A_1(r) = \pm \frac{Q}{rf}. \quad (30)$$

Equation (30) also corresponds to the $m \rightarrow \pm \frac{\sqrt{-\Lambda}}{2}$ limit of Eqs. (26) and (27).

V. THE OTHER SPECIFIC SOLUTIONS

In this section, we consider the cases where the temporal component of the vector field is not given by Eq. (23).

A. The solutions for the more general form of $A_0(r)$

First, we consider the case where the additional (inverse) power-law function of the radial coordinate r is added to

$A_0(r)$ shown in Eq. (23), namely,

$$A_0(r) = P + \frac{Q}{r} + Q_p r^p, \quad (31)$$

where p is the real number ($p \neq -1$) and Q_p is the constant.

For the temporal component of the vector field given by Eq. (31), the existence of the solution for an arbitrary p requires $\beta = \frac{1}{4}$, $P = \pm 2m_p$ and $m = \pm \frac{\sqrt{\Lambda}}{2}$ ($\Lambda > 0$).

For $p \neq -3, -\frac{3}{2}, -\frac{1}{2}$, the solution is given by

$$f(r) = \frac{1}{4m_p^2 r} \left[-8Mm_p^2 + \frac{4}{3}\Lambda m_p^2 r^3 + 4m_p^2 r \pm 4m_p Q_p r^{p+1} \left(\frac{\Lambda(p+1)r^2}{p+3} + 1 \right) + (p+1)^2 Q_p^2 r^{2p+1} \left(\frac{\Lambda r^2}{2p+3} + \frac{1}{2p+1} \right) \right], \quad (32a)$$

$$h(r) = \frac{1}{\left(1 \pm \frac{(p+1)Q_p r^p}{2m_p} \right)^2} f(r), \quad (32b)$$

$$\begin{aligned} A_1(r) = & \pm 2\sqrt{3}m_p \sqrt{(2p+1)(2p+3)(p+3)(\pm 2m_p + (p+1)Q_p r^p)} \\ & \times \left\{ -r \left[r \left(p^3 (16\Lambda m_p^2 r^2 \pm 48\Lambda m_p Q_p r^{p+2} + 3Q_p^2 r^{2p} (11\Lambda r^2 + 9)) \right. \right. \right. \\ & + p^2 (80\Lambda m_p^2 r^2 \pm 144\Lambda m_p Q_p r^{p+2} + 3Q_p^2 r^{2p} (19\Lambda r^2 + 9)) + 3\Lambda p r^2 (36m_p^2 \pm 44m_p Q_p r^p + 13Q_p^2 r^{2p}) \\ & + 9\Lambda r^2 (\pm 2m_p + Q_p r^p)^2 + 6p^4 Q_p^2 r^{2p} (\Lambda r^2 + 1)) - 24Mm_p^2 (2p+1)(2p+3)(p+3) \Big] \\ & + 6(2p+1)(2p+3)(p+3)Q_p (\pm 2m_p + Q_p r^p) + 3(2p+1)(2p+3)(p+3)Q_p^2 \Big\}^{\frac{1}{2}} \\ & \times \left\{ r \left[4m_p^2 (2p+1)(2p+3)(p+3)(\Lambda r^2 + 3) \pm 12m_p (2p+1)(2p+3)Q_p r^p (\Lambda p r^2 + p + \Lambda r^2 + 3) \right. \right. \\ & \left. \left. + 3(p+1)^2 (p+3)Q_p^2 r^{2p} (2p(\Lambda r^2 + 1) + \Lambda r^2 + 3) \right] - 24Mm_p^2 (2p+1)(2p+3)(p+3) \right\}^{-1}, \end{aligned} \quad (32c)$$

where the upper and lower branches of Eq. (32) correspond to $P = 2m_p$ and $P = -2m_p$, respectively. The point $r = \left[\mp \frac{2m_p}{(p+1)Q_p} \right]^{\frac{1}{p}}$ could be the curvature singularity other than at $r = 0$. For $P = 2m_p$, the appearance of the curvature singularity can be avoided for $Q_p < 0$ and $p < -1$ or for $Q_p > 0$ and $p > -1$, while for $P = -2m_p$ it can be avoided for $Q_p > 0$ and $p < -1$ or for $Q_p < 0$ and $p > -1$. For any

value of $p (\neq -3, -\frac{3}{2}, -\frac{1}{2})$ and $M > 0$, the function $f(r)$ has a single root which corresponds to the position of the unique event horizon. For a larger value of M , the singularity $r = \left[\mp \frac{2m_p}{(p+1)Q_p} \right]^{\frac{1}{p}}$ is hidden by the event horizon.

For the other values of $p = -3, -\frac{3}{2}, -\frac{1}{2}$, the similar solutions are obtained only for $\beta = \frac{1}{4}$ and $P = \pm 2m_p$. Here, we introduce the solution for each case:

1. For $p = -3$, the solution for $P = \pm 2m_p$ is given by

$$f(r) = \frac{1}{15m_p^2 r^6} \left[\pm 15m_p Q_{-3} r^3 - Q_{-3}^2 (3 + 5r^2 \Lambda) + 5m_p^2 r^5 (-6M + 3r + r^3 \Lambda) \mp 30m_p Q_{-3} r^5 \Lambda \ln(r) \right], \quad (33a)$$

$$h(r) = \frac{m_p^2 r^6}{(Q_{-3} \mp m_p r^3)^2} f(r), \quad (33b)$$

$$A_1(r) = \pm \frac{\sqrt{15}m_p (Q_{-3} \mp m_p r^3)}{\pm 15m_p Q_{-3} r^3 - Q_{-3}^2 (3 + 5r^2 \Lambda) + 5m_p^2 r^5 (-6M + 3r + r^3 \Lambda) \mp 30m_p Q_{-3} r^5 \Lambda \ln(r)} \\ \times \left\{ 30Q_{-3} r^2 + Q_{-3}^2 (27 + 20r^2 \Lambda) + 5r^4 (3Q^2 \pm 12(\pm 2Mm_p + Q)m_p r - 4m_p^2 r^4 \Lambda) \pm 120m_p Q_{-3} r^5 \Lambda \ln(r) \right\}^{\frac{1}{2}}. \quad (33c)$$

The point $r = \left(\pm \frac{Q_{-3}}{m_p} \right)^{\frac{1}{3}}$ could be the curvature singularity other than at $r = 0$, which is absent for $Q_{-3} < 0$ in the positive branch and for $Q_{-3} > 0$ in the negative branch.

2. For $p = -\frac{3}{2}$, the solution for $P = \pm 2m_p$ is given by

$$f(r) = \frac{1}{96m_p^2 r^3} \left[-3Q_{-3/2}^2 \mp 32m_p Q_{-3/2} r^{\frac{3}{2}} (-3 + r^2 \Lambda) + 32m_p^2 r^2 (-6M + r(3 + r^2 \Lambda)) + 6Q_{-3/2}^2 r^2 \Lambda \ln(r) \right], \quad (34a)$$

$$h(r) = \frac{16m_p^2 r^3}{\left(Q_{-3/2} \mp 4m_p r^{\frac{3}{2}} \right)^2} f(r), \quad (34b)$$

$$A_1(r) = \pm \frac{2\sqrt{6}m_p (Q_{-3/2} \mp 4m_p r^{3/2})}{-3Q_{-3/2}^2 \mp 32m_p Q_{-3/2} r^{\frac{3}{2}} (-3 + r^2 \Lambda) + 32m_p^2 r^2 (-6M + r(3 + r^2 \Lambda)) + 6Q_{-3/2}^2 r^2 \Lambda \ln(r)} \\ \times \left\{ 8r(24m_p^2 M r - 4\Lambda m_p^2 r^4 \pm 12m_p Q r + 3Q^2) + 16Q_{-3/2} \sqrt{r} (\pm 2\Lambda m_p r^3 + 3Q) - 6\Lambda Q_{-3/2}^2 r^2 \ln(r) + 27Q_{-3/2}^2 \right\}^{\frac{1}{2}}. \quad (34c)$$

The point $r = \left(\pm \frac{Q_{-3/2}}{4m_p} \right)^{\frac{2}{3}}$ could be the curvature singularity other than at $r = 0$, which is absent for $Q_{-3/2} < 0$ in the positive branch and for $Q_{-3/2} > 0$ in the negative branch.

3. For $p = -\frac{1}{2}$, the solution for $P = \pm 2m_p$ is given by

$$f(r) = \frac{1}{480m_p^2 r} \left[-960Mm_p^2 + 15Q_{-1/2}^2 r^2 \Lambda + 160m_p^2 r (3 + r^2 \Lambda) \pm 96m_p Q_{-1/2} \sqrt{r} (5 + r^2 \Lambda) + 30Q_{-1/2}^2 \ln(r) \right], \quad (35a)$$

$$h(r) = \frac{16m_p^2 r}{\left(Q_{-1/2} \pm 4m_p r^{\frac{1}{2}} \right)^2} f(r), \quad (35b)$$

$$A_1(r) = \pm \frac{2\sqrt{30}m_p (Q_{-1/2} \pm 4m_p r^{\frac{1}{2}})}{\sqrt{r} \left[-960Mm_p^2 + 15Q_{-1/2}^2 r^2 \Lambda + 160m_p^2 r (3 + r^2 \Lambda) \pm 96m_p Q_{-1/2} \sqrt{r} (5 + r^2 \Lambda) + 30Q_{-1/2}^2 \ln(r) \right]} \\ \times \left\{ 960Mm_p^2 r - 160\Lambda m_p^2 r^4 + 240Q(\pm 2m_p r + Q_{-1/2} \sqrt{r}) \mp 96\Lambda m_p Q_{-1/2} r^{7/2} + 120Q^2 \right. \\ \left. - 15\Lambda Q_{-1/2}^2 r^3 + 120Q_{-1/2}^2 r - 30Q_{-1/2}^2 r \ln(r) \right\}^{\frac{1}{2}}. \quad (35c)$$

The point $r = \left(\mp \frac{Q_{-1/2}}{4m_p} \right)^2$ could be the curvature singularity other than at $r = 0$, which is absent for $Q_{-1/2} > 0$ in the positive branch and for $Q_{-1/2} < 0$ in the negative branch.

For all the values of $p = -3, -\frac{3}{2}, -\frac{1}{2}$, the functions $f(r)$ and $h(r)$ have a single root which corresponds to the position

of the unique event horizon. In the limit of $\Lambda \rightarrow 0$ and $Q_p \rightarrow 0$, the solutions (32), (33), (34) and (35) reduce to the stealth Schwarzschild solution (25) with $P = \pm 2m_p$.

Moreover, as argued in Ref. [36], only for $m = 0$, $\beta = \frac{1}{4}$ and $p = 2$ in Eq. (31), the other type of the solution can be obtained for

$$Q_2 = \frac{2m_p^2 P \Lambda}{3(P^2 - 4m_p^2)}, \quad (36)$$

given by

$$f(r) = 1 - \frac{2M}{r} + \frac{4m_p^2 r^2 \Lambda (5P^2 + m_p^2 (-20 + 3r^2 \Lambda))}{15(P^2 - 4m_p^2)^2}, \quad (37a)$$

$$h(r) = \frac{(P^2 - 4m_p^2)^2}{(P^2 + 2m_p^2(-2 + r^2 \Lambda))^2} f(r), \quad (37b)$$

$$A_1(r) = \pm \sqrt{\frac{A_0(r)^2}{f(r)h(r)} - \frac{P^2 + 2m_p^2 r^2 \Lambda}{h(r)}}. \quad (37c)$$

The spacetime is neither asymptotically Minkowski nor (anti-) de Sitter. If $\frac{4m_p^2 - P^2}{\Lambda} > 0$, the point $r = \frac{1}{m_p} \sqrt{\frac{4m_p^2 - P^2}{2\Lambda}}$ is the curvature singularity other than $r = 0$. On the other hand, if $\frac{4m_p^2 - P^2}{\Lambda} < 0$, there is no curvature singularity except for $r = 0$. In both the cases, the function $f(r)$ always has a single root which corresponds to the position of the unique event horizon, and for $\frac{4m_p^2 - P^2}{\Lambda} > 0$ the singularity at $r = \frac{1}{m_p} \sqrt{\frac{4m_p^2 - P^2}{2\Lambda}}$ is hidden by the event horizon $\sqrt{\Lambda} M > \frac{4}{15m_p} \sqrt{\frac{4m_p^2 - P^2}{2}}$. For the other values of p the analytic solution similar to Eq. (37) could not be found.

B. The solution for $A_1(r) = 0$

So far, we have investigated the static and spherically symmetric solutions for several choices of $A_0(r)$. Instead, we may specify $A_1(r)$ and then find the other variables $f(r)$, $h(r)$ and $A_0(r)$ by solving the equations of motion (4a) and (4b) under the ansatz (9) and (10). For $m \neq 0$ and/or $\beta \neq 0$, the r component of the vector field equation of motion (4b) becomes nontrivial as

$$0 = h(r)A_1(r) \times [-m^2 r^2 f(r) + \beta(-f(r)(1 - h(r)) + rh(r)f'(r))], \quad (38)$$

which as in general $h(r) \neq 0$ gives the two possibilities

$$-m^2 r^2 f(r) + \beta(-f(r)(1 - h(r)) + rh(r)f'(r)) = 0, \quad (39a)$$

or

$$A_1(r) = 0. \quad (39b)$$

The solutions, Eqs. (14), (17), (21), (25), (26), (27), (28), (32), (33), (34), (35), and (37), have been originated from the former choice (39a). Under the ansatz Eqs. (9) and (10), if the r component of the vector field equation of motion (38) is satisfied then the (t, r) component of the Einstein equation is also automatically satisfied.

For the latter case (39b), the specific solution is obtained only for $m = 0$, $\Lambda = 0$ and $\beta = \frac{1}{4}$ by

$$f(r) = h(r) = 1 \pm \sqrt{\frac{r_0}{r}}, \quad A_0(r) = 2m_p f(r), \quad (40)$$

where r_0 is the integration constant. The solution (40) was obtained in Ref. [38]. The spacetime is asymptotically flat. The singularity at $r = 0$ is visible in the positive branch and hidden by the event horizon at $r = r_0$ in the negative branch, respectively. The solution similar to Eq. (40) could not be found for the more general cases of $m \neq 0$, $\Lambda \neq 0$, or $\beta \neq \frac{1}{4}$.

C. A short summary

Throughout Secs. IV and V, we have obtained the static and spherically symmetric solutions for several nontrivial choices of the temporal component of the vector field $A_0(r)$.

For $A_0(r)$ with the form of the Coulomb potential given by Eq. (23), we have obtained the stealth Schwarzschild, the Schwarzschild- (anti-) de Sitter and the asymptotically anti-de Sitter solutions (25), (26) and (27), and (28), respectively. Unexpectedly, these solutions are present only for the specific value of the nonminimal coupling constant, $\beta = \frac{1}{4}$, and the electric charge Q does not appear in the metric, which is different from the case of the Reissner- Nordström [- (anti-) de Sitter] solution (24).

For the other cases, we could obtain the solutions (32), (33), (34), (35), (37) and (40). All these solutions also exist only for $\beta = \frac{1}{4}$.

VI. THE SLOWLY ROTATING SOLUTIONS

Finally, we investigate the slowly rotating solutions within the Hartle-Thorne approximation [41, 42], where the rotational correction is obtained in the perturbation framework to the static and spherically symmetric background with respect to the angular velocity of the black hole Ω .

The correction to the static and spherically symmetric metric (9) appears in the (t, ϕ) component at $\mathcal{O}(\Omega)$ and in the other components at $\mathcal{O}(\Omega^2)$. Thus within $\mathcal{O}(\Omega)$ the metric will take the form of

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{h(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) - 2r^2 \omega(r) \sin^2 \theta dt d\phi, \quad (41)$$

where $\omega(r)$ is the unknown function of $\mathcal{O}(\Omega)$. Similarly, the correction to the vector field in the static and spherically symmetric background (10) appears in the ϕ component at $\mathcal{O}(\Omega)$

and in the other components at $\mathcal{O}(\Omega^2)$. Hence, within $\mathcal{O}(\Omega)$ the vector field will take the form of

$$A_\mu = (A_0(r), A_1(r), 0, A_3(r, \theta)). \quad (42)$$

For the separability of the equations of motion at $\mathcal{O}(\Omega)$, we assume that the azimuthal component of the vector field $A_3(r, \theta)$ takes the form

$$A_3(r, \theta) = a_3(r) \sin^2 \theta, \quad (43)$$

where $a_3(r)$ is the other unknown function of $\mathcal{O}(\Omega)$.

The unknown functions $\omega(r)$ and $a_3(r)$ in Eqs. (41) and (43) are found as the solution of the field equations (4a) and (4b) at $\mathcal{O}(\Omega)$. At $\mathcal{O}(\Omega)$, only the (t, ϕ) component of the Einstein equation (4a) becomes nontrivial, and similarly only the ϕ component of the vector field equation of motion (4b) becomes nontrivial. Thus they will be solved under the boundary conditions that $\omega(r)$ and $a_3(r)$ are finite in the large- r limit.

Before starting, we consider the case of $m = \beta = 0$ where the gauge symmetry is recovered. In this case, the slow-rotation correction to the Reissner-Nörsröm-(anti-) de Sitter solution (24) is obtained by

$$\omega(r) = \omega_0 + \frac{2J}{r^3} - \frac{JQ^2}{2m_p^2 M r^4}, \quad (44a)$$

$$a_3(r) = -\frac{JQ}{Mr}, \quad (44b)$$

which agrees with the Kerr-Newman-(anti-) de Sitter solution by neglecting the terms of $\mathcal{O}(\Omega^2)$ [45, 46].

A. For the background with the constant $A_0(r)$

First, we consider the background solutions with the constant temporal component of the vector field, $A_0(r) = P$ discussed in Sec. III, namely, Eqs. (14) and (17) for $P \neq 0$ and Eq. (21) for $P = 0$.

For these background solutions, if we assume that $a_3(r) = 0$, we find that $\omega(r)$ remains the same as the slow-rotation limit of the Schwarzschild-(anti-) de Sitter background in GR, given by

$$\omega(r) = \omega_0 + \frac{2J}{r^3}, \quad (45)$$

where the constant $\omega_0 = 0$ for the Schwarzschild background and $\omega_0 \neq 0$ for the Schwarzschild-(anti-) de Sitter background, and the constant J represents the angular momentum of the black hole. This is the confirmation of the argument in Sec. III at the level of the first order in the slow-rotation approximation, $\mathcal{O}(\Omega)$, as the solutions in the scalar-tensor theory (8) obtained in Refs. [43, 44] are expressed as those in the generalized Proca theory with the vanishing field strength.

On the other hand, if we assume that $a_3(r) \neq 0$, formally the more general solution can be found. For instance, for the stealth Schwarzschild background (14) with $P = \pm \frac{m_p}{\sqrt{\beta}}$ ($\beta >$

0), the solution is given by

$$\omega(r) = \omega_0 + \frac{2J}{r^3} \mp \frac{3QM}{2m_p \sqrt{\beta} r^4}, \quad (46a)$$

$$a_3(r) = \frac{Q}{r}, \quad (46b)$$

where Q is the integration constant. The same solution as Eq. (46) is also obtained for the Schwarzschild-(anti-) de Sitter background (17) $m = \pm \sqrt{\beta \Lambda}$ ($\beta \Lambda > 0$). For the other background parameters, we could obtain the solutions with the same leading behavior as Eq. (46) in the large- r limit. From Eq. (46a), the contribution of Q seems to appear as an independent “charge”.

In fact, if we consider the slow-rotation correction to the Schwarzschild-(anti-) de Sitter solution with $m = \beta = 0$ by assuming that $a_3(r) \neq 0$, the solution with the same leading behavior in the large- r limit as Eq. (46b) could be obtained. However, we have to set the integration constant $Q = 0$, as the slow-rotation correction to the electrically neutral background solution could not induce the nonzero magnetic field. Similarly in our case, for the background with the vanishing electric field strength it is reasonable to set $Q = 0$, and hence Eq. (45) with $a_3(r) = 0$ can be regarded as the only physical solution.

B. For the background with the nonconstant $A_0(r)$

Second, we consider the background solutions with the nonconstant $A_0(r)$ discussed in Sec. IV, namely Eqs. (25), (26), (27) and (28) with $m = \pm \frac{\sqrt{-\Lambda}}{2}$ where the solution reduces to Eq. (30). We find that $\omega(r)$ is given by Eq. (45) which is the same as the Kerr-(anti-) de Sitter solution, and $a_3(r)$ is given by

$$a_3(r) = -\frac{JQ}{Mr}, \quad (47)$$

which is similar to the result obtained in Ref. [36] for the stealth Schwarzschild background (25). Thus $a_3(r)$ is the same as the slow-rotation limit of the Kerr-Newman-(anti-) de Sitter solution (44b), but $\omega(r)$ does not contain the term depending on the background electric charge Q . This is the stealth property realized at the first order in the slow-rotation approximation, $\mathcal{O}(\Omega)$. For Eq. (28) with $m \neq \pm \frac{\sqrt{-\Lambda}}{2}$, no analytic solution of $\omega(r)$ and $a_3(r)$ could be obtained.

C. A short summary

In this section, we have investigated the slow-rotation corrections to the static and spherically symmetric backgrounds within the first order of the Hartle-Thorne approximation [41, 42].

For the background with the vanishing electric field strength, the slow-rotation correction to the metric was found to be the same as the Kerr-(anti-) de Sitter solution (45) with $A_3(r, \theta) = 0$.

On the other hand, for the background with the nonvanishing electric field strength, the slow-rotation correction to the metric remains the same as the Kerr- (anti-) de Sitter solution (45), but the azimuthal component $A_3(r, \theta)$ is the same as the Kerr- Newman- (anti-) de Sitter solution (47), which is the realization of the stealth property in the slowly rotating case.

VII. CONCLUSIONS

We have investigated the static and spherically symmetric solutions in the generalized Proca theory with the nonminimal coupling of the vector field to the Einstein tensor (1). First, we have shown that the solutions obtained in the scalar-tensor theory with the nonminimal derivative coupling to the Einstein tensor (8) are also those in the generalized Proca theory (1) with the vanishing field strength, and we have obtained the expressions of the stealth Schwarzschild, the Schwarzschild-(anti-) de Sitter and the asymptotically anti- de Sitter solutions in the generalized Proca theory.

Second, we have investigated these solutions where the temporal component of the vector field contains the term of the Coulomb potential. In this case, as argued in Ref. [36], the extension of these solutions requires the special value of the nonminimal coupling parameter, irrespective of the value of the mass term of the vector field and the asymptotic property of the spacetime. We have also obtained the other nontrivial solutions for the same value of the coupling constant.

Finally, we have investigated the first-order slow-rotation corrections to the static and spherically symmetric solutions.

We have found that for the background with the vanishing electric field strength the slowly rotating solution remains the same as in GR. For the background with the nonvanishing electric field strength, the slow-rotation correction to the metric does not depend on the electric charge and may be regarded as the realization of the stealth property in the context of the slow-rotation approximation.

There will be various extensions of the present work. The first subject is to investigate the stability of the solutions obtained in this paper. The stability of the black hole solutions in the scalar-tensor Horndeski theory (8) has been investigated in Refs. [44, 47–49]. Especially in Ref. [49] it was argued that the static and spherically symmetric black hole solutions with the constant canonical kinetic term $X_\varphi = -\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$ are generically unstable in the vicinity of the event horizon. It will be interesting to investigate whether there is the same kind of instability in the vector-tensor theory. The other interesting issue is to investigate the rapidly rotating black hole solutions and the spectrum of the quasinormal modes, which could make the deviations from the GR solutions more evident. Other than the vacuum solutions, it will also be very important to investigate the solutions of the neutron stars and the other compact objects. (See, e.g., [50–53] for the related studies in the case of the scalar-tensor Horndeski theory.) We hope to come back to these issues in future work.

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